ASTRON 329/429, Fall 2017 – Problem Set 3

Due on Wed. Nov 8, by 5 PM in Alex Gurvich's mailbox.

Solve the following problems, plus problems 7.1, 7.2, 7.3 in Ryden.

I. Einstein radius. Consider a point gravitational lens of mass M perfectly aligned between a source and an observer. Let d be the distance between the source, and let xd be the distance between the lens and the observer. You may assume that the source is sufficiently nearby that d is an ordinary distance (i.e., neglect cosmological expansion). Derive the expression for the Einstein radius given in class,

$$\theta_{\rm E} = \left(\frac{4GM}{c^2 d} \frac{1-x}{x} \right)^{1/2} \tag{1}$$

(eq. 7.45 in Ryden).

II. Free fall times. The free fall time is the characteristic time that it takes a body to fall into a medium of mean density $\bar{\rho}$ under the action of gravity (for example, the time it would take a galaxy to fall to the center of a galaxy cluster absent support of its orbit by angular momentum). It is also sometimes called the dynamical time. For a spherically symmetric, homogeneous distribution of matter the free fall time can be shown to be

$$t_{\rm ff} = \left(\frac{3\pi}{32 G \bar{\rho}} \right)^{1/2} \approx 0.5 \frac{1}{\sqrt{G\bar{\rho}}}. \tag{2}$$

We have seen in class that the age of the Universe at any redshift can be estimated as the Hubble time $t_{\rm H} = H(z)^{-1}$.

a) For a critical universe, show that at any redshift, the free fall time at the mean (total) density of the universe $\rho_{\rm crit}$ is approximately equal to the age of the universe at that redshift, i.e. $t_{\rm ff}(\rho_{\rm crit}(z)) \sim t_{\rm H}(z)$ for any z.

b) Dark matter halos are defined as structures whose mean enclosed density is 200 times the critical density, i.e. $\bar{\rho}_{\rm halo}(z) = 200\rho_{\rm crit}(z)$. Find a general expression, valid for halos of any mass at any redshift, expressing the halo free fall time as a function of $t_{\rm H}$.